

MATROIDS REPRESENTABLE OVER FIELDS WITH A COMMON SUBFIELD

PETER NELSON, STEFAN H.M. VAN ZWAM

ABSTRACT. A matroid is $\text{GF}(q)$ -*regular* if it is representable over all proper superfields of the field $\text{GF}(q)$. We show that, for highly connected matroids having a large projective geometry over $\text{GF}(q)$ as a minor, the property of $\text{GF}(q)$ -regularity is equivalent to representability over both $\text{GF}(q^2)$ and $\text{GF}(q^t)$ for some odd integer $t \geq 3$. We do this by means of an exact structural description of all such matroids.

1. INTRODUCTION

For a field \mathbb{F}_0 , we say a matroid M is \mathbb{F}_0 -*regular* if M is representable over every field \mathbb{F} having \mathbb{F}_0 as a proper subfield.

Let $n \geq 2$ be an integer, q be a prime power, and N be a $\text{PG}(n-1, q)$ -restriction of a matroid $M \cong \widehat{\text{PG}}(n-1, q^2)$. Let L_0 be a line of N and $x \in \text{cl}_M(L_0) - L_0$. We denote by $\widehat{\text{PG}}(n-2, q)$ any matroid isomorphic to $\text{si}((M/x)|E(N))$. If $n \geq 3$ and $f \in E(N) - L_0$, then we denote by $\overline{\text{PG}}(n-1, q)$ any matroid isomorphic to $M|(E(N) \cup \text{cl}_M(\{x, f\}))$. (We will show later that these matroids are uniquely determined up to isomorphism.) A matroid M is *round* if $E(M)$ is not the union of two hyperplanes, or equivalently if M is infinitely vertically connected. Our main theorem is the following:

Theorem 1.1. *Let q be a prime power and M be a round rank- r matroid with a $\text{PG}(12q^{12} + 19, q)$ -minor. The following are equivalent:*

- (1) M is $\text{GF}(q)$ -regular;
- (2) M is representable over $\text{GF}(q^2)$ and $\text{GF}(q^t)$ for some odd integer $t \geq 3$; and
- (3) $\text{si}(M)$ is a restriction of either $\widehat{\text{PG}}(r-1, q)$ or $\overline{\text{PG}}(r-1, q)$.

This exactly characterises all $\text{GF}(q)$ -regular matroids that are sufficiently ‘rich’ and highly connected; the equivalence of (1) and (2) is strongly reminiscent of Tutte’s characterisation of regular matroids of the usual sort, and motivates our use of the word. This equivalence

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may hold for all matroids (this has essentially been conjectured for $q = 2$ in [9, Conjecture 6.8]), but the characterisation in (3) requires some extra hypotheses, and we briefly discuss the ones we chose.

As one could otherwise construct counterexamples by taking 2-sums and 3-sums, some connectivity assumption is needed. However, the hypothesis of roundness is probably overkill. The theorem likely holds for vertically 4-connected matroids, and many of our techniques apply in this more general setting. Proving a ‘vertically 4-connected’ version of the theorem would require analysis of how the structure in (3) propagates over 4-separations.

The hypothesis of having some sort of underlying ‘richness’, here a large projective geometry minor, is also necessary; the structure in (3) does not describe all vertically 4-connected $\text{GF}(q)$ -regular matroids. Indeed, Gerards [6] defined a class of signed-graphic matroids representable over every field with at least three elements; this class contains counterexamples to our theorem of arbitrarily high branch-width. However, Gerards’ counterexamples are nearly planar; it is possible that a very similar structure to that in (3) holds for all vertically 4-connected matroids with a large enough clique minor. Round $\text{GF}(q^2)$ -representable matroids of huge rank have a large clique minor [4], so in the round setting it is possible that our hypothesis of a large projective geometry minor could be replaced with a ‘large rank’ hypothesis with few other changes to the theorem statement.

Though the material in this paper is self-contained, sections 6 and 7 make essential use of the theory of tangles and some currently unpublished techniques due to Geelen, Gerards and Whittle [5].

2. PRELIMINARIES

We largely follow the notation of Oxley [8]. We also write $\epsilon(M)$ for $|\text{si}(M)|$. For a positive integer n , we denote the set $\{1, \dots, n\}$ by $[n]$. Finally, if \mathbb{F}_0 is a subfield of a field \mathbb{F} and A is an \mathbb{F} matrix, we write $\text{row}_{\mathbb{F}_0}(A)$ for the vector space containing all linear combinations of the rows of A with coefficients in \mathbb{F}_0 . We define $\text{col}_{\mathbb{F}_0}(A)$ similarly.

The versions of connectivity we consider are all ‘vertical’; for $k \in \mathbb{Z}^+ \cup \{\infty\}$ a set $A \subseteq E(M)$ is *vertically k -separating* in M if $\lambda_M(A) < k$ and $\min(r_M(A), r(M \setminus A)) \geq k$, and M is *vertically k -connected* if M has no vertically k' -separating subsets for any $k' \leq k$. M is *round* if it is vertically ∞ -connected; for example cliques, projective geometries and non-binary affine geometries are round. A matroid M is vertically k -connected if and only if its simplification is vertically k -connected.

Moreover if M is vertically k -connected then M/e is vertically $(k-1)$ -connected for each $e \in E(M)$; in particular if M is round then so is M/e . We will use the following slight strengthening of a well-known result on connectivity; see [8, Theorem 8.5.7].

Theorem 2.1 (Tutte's Linking Theorem). *Let M be a matroid and $A, B \subseteq E(M)$ be disjoint sets. There is a minor N of M so that $E(N) = A \cup B$, $N|A = M|A$, $N|B = M|B$ and $\lambda_N(A) = \kappa_M(A, B)$.*

To avoid complications arising from inequivalent representations, we will often consider matroids defined by a representation rather than axiomatically. If \mathbb{F} is a field, then an \mathbb{F} -represented matroid on ground set E is a pair $M = (U, E)$, where U is a subspace of \mathbb{F}^E . This represented matroid has rank function given by $r_M(X) = \dim(U[X])$ for each $X \subseteq E$, where $U[X]$ is the projection of U onto \mathbb{F}^X . Where confusion might arise, we refer to a matroid defined in the usual way as an *abstract* matroid; if M is an \mathbb{F} -represented matroid then we write \tilde{M} for the abstract matroid with the same rank function as M .

Given a matrix $A \in \mathbb{F}^{X \times E}$, we write $M(A)$ for the \mathbb{F} -represented matroid $(\text{row}(A), E)$ and $\tilde{M}(A)$ for the associated abstract matroid; here A is an \mathbb{F} -representation of $M(A)$. We also need to formalize deletion and contraction in this context; given an \mathbb{F} -representation A of an \mathbb{F} -represented matroid M and a set $X \subseteq E(M)$, we write $M \setminus X$ for the \mathbb{F} -represented matroid $M(A[E(M) - X])$. It is easiest to define contraction in terms of duality; if $M = (U, E)$ is an \mathbb{F} -represented matroid then let $M^* = (U^\perp, E)$, where $U^\perp = \{v \in \mathbb{F}^E : \langle v, u \rangle = 0 \text{ for all } u \in U\}$, and $M/X = (M^* \setminus X)^*$. Given a particular representation A , this is equivalent to the usual matrix interpretation of contraction where we row-reduce and take a submatrix of A . We extend these definitions to define a *minor* and *restriction* of an \mathbb{F} -represented matroid, as well as extending all other usual matroidal notions such as connectivity.

If \mathbb{F}_0 is a subfield of \mathbb{F} , then two \mathbb{F} -matrices A_1, A_2 are \mathbb{F}_0 -row-equivalent if one can be obtained from the other by elementary row-operations only involving coefficients in \mathbb{F}_0 . Furthermore, the matrices A_1, A_2 are \mathbb{F}_0 -projectively equivalent if there is a matrix A'_1 that is \mathbb{F}_0 -row-equivalent to A_1 that can be obtained from A_2 by scaling columns by nonzero elements of \mathbb{F}_0 . We also say that the \mathbb{F} -represented matroids $M(A_1)$ and $M(A_2)$ are \mathbb{F}_0 -projectively equivalent. If $\mathbb{F}_0 = \mathbb{F}$ then we just say the matrices or represented matroids are *projectively equivalent*, and write $A_1 \approx A_2$ and $M(A_1) \approx M(A_2)$. It is clear that if $M \approx M'$ then $\tilde{M} = \tilde{M}'$. For each integer n , let $\mathcal{PG}(n-1, q)$ denote the set of GF(q)-matrices G with row-set $[n]$ satisfying $\tilde{M}(G) \cong \text{PG}(n-1, q)$.

3. ALGEBRA

We frequently consider an extension field \mathbb{F} of a field \mathbb{F}_0 ; our main theorem applies just when $\mathbb{F}_0 = \text{GF}(q)$ and $\mathbb{F} = \text{GF}(q^2)$, but some lemmas apply for arbitrary \mathbb{F}_0 . When the extension has degree 2 with $\mathbb{F} = \mathbb{F}_0(\omega)$, we often use the fact that \mathbb{F} is a dimension-2 vector space over \mathbb{F}_0 with basis $\{1, \omega\}$. We require a few lemmas relating \mathbb{F}_0 and \mathbb{F} in various contexts; the first is proved in [7].

Lemma 3.1. *Let $n \geq 3$ be an integer, q be a prime power, and \mathbb{F} be a field with a $\text{GF}(q)$ -subfield. If A is an \mathbb{F} -matrix with $M(A) \cong \text{PG}(n-1, q)$, then A is projectively equivalent to a $\text{GF}(q)$ -matrix.*

We will apply the next lemma in the case where $j = 2$ and $h = 3$.

Lemma 3.2. *Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field \mathbb{F}_0 and let $j, h, t \in \mathbb{Z}^+$ satisfy $2j > h$ and $j, h \leq t$. If V is an h -dimensional subspace of \mathbb{F}_0^t and U is a j -dimensional subspace of \mathbb{F}^t such that $U \subseteq \text{span}_{\mathbb{F}}(V)$, then $U \cap V$ is nontrivial.*

Proof. Let $\{b_1, \dots, b_h\}$ be a basis for V and let $W = \text{span}_{\mathbb{F}}(V)$, noting that each $w \in W$ is expressible in the form $\sum_{i=1}^h (\lambda_i + \omega \mu_i) b_i$ for some unique $\lambda, \mu \in \mathbb{F}_0^h$. Let $\varphi : W \rightarrow \mathbb{F}_0^{2h}$ be the invertible linear transformation defined by $\varphi\left(\sum_{i=1}^h (\lambda_i + \omega \mu_i) b_i\right) = (\lambda_1, \dots, \lambda_h, \mu_1, \dots, \mu_h)$. Now $\varphi(U)$ and $\varphi(V)$ are subspaces of \mathbb{F}_0^{2h} with $\dim(\varphi(U)) = 2j$ and $\dim(\varphi(V)) = h$, so $\dim(\varphi(U) \cap \varphi(V)) = 2j + h - 2h > 0$. Therefore $U \cap V$ is nontrivial, as required. \square

Lemma 3.3. *Let \mathbb{F}_0 be a field and $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of \mathbb{F}_0 . Let $h, d, n \in \mathbb{Z}_0^+$ satisfy $h \leq d$ and let $A, B \in \mathbb{F}_0^{d \times n}$ be matrices such that $\text{rank}(A + \omega B) = d$. If $\text{rank}\begin{pmatrix} A \\ B \end{pmatrix} = 2d - h$ then there is a rank- h matrix $Q \in \mathbb{F}^{h \times d}$ such that $Q(A + \omega B)$ is an \mathbb{F}_0 -matrix.*

Proof. Let $\omega^2 = s + \omega t$ for $s, t \in \mathbb{F}_0$. If $\text{rank}\begin{pmatrix} A \\ B \end{pmatrix} = 2d - h$ then there are matrices $Q_1, Q_2 \in \mathbb{F}_0^{h \times d}$ such that $(Q_1 | Q_2) \begin{pmatrix} A \\ B \end{pmatrix} = Q_1 A + Q_2 B = 0$ and $\text{rank}(Q_1 | Q_2) = h$. Let $Q = (\omega - t)Q_1 + Q_2$; we have $Q(A + \omega B) = (Q_2 A - tQ_1 A + sQ_1 B) + \omega(Q_1 A + Q_2 B)$ which is an \mathbb{F}_0 -matrix.

It remains to show that $\text{rank}(Q) = h$. If not, then there are row vectors $x, y \in \mathbb{F}_0^h$ such that $x + \omega y \neq 0$ and $(x + \omega y)Q = 0$. This gives $(xQ_2 - txQ_1 + syQ_1) + \omega(xQ_1 + yQ_2) = 0$, implying that

$$(1) \quad \begin{pmatrix} -t & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} Q_1 + \begin{pmatrix} x \\ y \end{pmatrix} Q_2 = 0.$$

Note that the matrix $J = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}$ satisfies $\begin{pmatrix} 0 & s \\ 1 & t \end{pmatrix} J = J \begin{pmatrix} t & -s \\ -1 & 0 \end{pmatrix}$. Set $\begin{pmatrix} u \\ v \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix} Q_1$; we will argue that $u + \omega v \neq 0$ and $(u + \omega v)(A + \omega B) = 0$,

which contradicts $\text{rank}(A + \omega B) = d$. If $u + \omega v = 0$, then $\begin{pmatrix} u \\ v \end{pmatrix} = 0$ and, since J is nonsingular, $\begin{pmatrix} x \\ y \end{pmatrix} Q_1 = 0$. This implies $xQ_1 = yQ_1 = 0$, which together with (1) and the fact that $\text{rank}(Q_1|Q_2) = h$ yields $\begin{pmatrix} x \\ y \end{pmatrix} = 0$, which is not the case. Therefore $\begin{pmatrix} u \\ v \end{pmatrix} \neq 0$. We have $(u + \omega v)(A + \omega B) = (uA + svB) + \omega(uB + vA + tvB) = \langle \begin{pmatrix} 1 \\ \omega \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} A + \begin{pmatrix} 0 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} B \rangle$. Now

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} A + \begin{pmatrix} 0 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} B &= J \begin{pmatrix} x \\ y \end{pmatrix} Q_1 A + \begin{pmatrix} 0 & s \\ 1 & t \end{pmatrix} J \begin{pmatrix} x \\ y \end{pmatrix} Q_1 B \\ &= J \left(\begin{pmatrix} x \\ y \end{pmatrix} Q_1 A + \begin{pmatrix} t & -s \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} Q_1 B \right) \\ &= -J \left(\begin{pmatrix} x \\ y \end{pmatrix} Q_2 + \begin{pmatrix} -t & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} Q_1 \right) B, \end{aligned}$$

since $Q_1 A = -Q_2 B$. Now combining the above with (1) we see that $(u + \omega v)(A + \omega B) = 0$, contradicting the fact that $\text{rank}(A + \omega B) = d$ and $u + \omega v \neq 0$. \square

The above lemma has the following as a straightforward corollary.

Lemma 3.4. *Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field \mathbb{F}_0 . Let $h, d, m, n \in \mathbb{Z}_0^+$ satisfy $0 \leq h \leq d \leq n$ and $A, B \in \mathbb{F}_0^{d \times n}$ and $P \in \mathbb{F}_0^{m \times n}$ be such that $\text{rank} \begin{pmatrix} A + \omega B \\ P \end{pmatrix} = m + d$, $\text{rank}(P) = m$ and $\text{rank} \begin{pmatrix} A \\ B \\ P \end{pmatrix} \leq m + 2d - h$. There exist matrices $A', B' \in \mathbb{F}_0^{d \times n}$ such that $\begin{pmatrix} A + \omega B \\ P \end{pmatrix}$ and $\begin{pmatrix} A' + \omega B' \\ P \end{pmatrix}$ are row-equivalent and B' has h zero rows.*

4. EXAMPLES

We now investigate the two classes of GF(q)-regular matroids from our main theorem. We define them differently from in the introduction in order to prove that they are both well-defined and GF(q)-regular. We will use the fact that projective geometries are *modular*; that is, that every pair of flats F_1, F_2 satisfies $r(F_1 \cap F_2) = r(F_1) + r(F_2) - r(F_1 \cup F_2)$.

Let \mathbb{F} be a field with a GF(q)-subfield, $n \geq 3$ be an integer, $A \in \mathcal{PG}(n-1, q)$ and $N = \tilde{M}(A) \cong \text{PG}(n-1, q)$. Let L_0 be a line of N and $v \in \text{col}_{\mathbb{F}}(A[L_0])$ be not parallel to any column of $A[L_0]$. Let $f \in E(N) - L_0$ and \mathcal{L} be the collection of lines of $\text{cl}_N(L_0 \cup \{f\})$ not containing f , noting that $|\mathcal{L}| = q^2$. For each $L \in \mathcal{L}$, let v_L be a nonzero vector in the rank-1 subspace $\text{col}_{\mathbb{F}}(A_L) \cap \text{col}_{\mathbb{F}}(v|A[f])$. Let $X = \{x_L : L \in \mathcal{L}\}$ be a q^2 -element set and let $\bar{A} \in \mathbb{F}^{[n] \times (E(N) \cup X)}$ be the matrix so that $\bar{A}[E(N)] = A$ and $\bar{A}[x_L] = v_L$ for each $L \in \mathcal{L}$.

Lemma 4.1. *The matroid $\tilde{M}(\overline{A})$ is determined up to isomorphism by the choice of n and q .*

Proof. Let $M = \tilde{M}(\overline{A})$. We have $M \setminus X = N \cong \text{PG}(n-1, q)$. Let \mathcal{F}_N be the set of cyclic flats of N and \mathcal{F}_M be that of M . Let $P = \text{cl}_N(L_0 \cup \{f\})$. Note that every pair of lines of P intersect. It is easy to check the following claim:

4.1.1.

$$\begin{aligned} \mathcal{F}_M = & \{F : F \in \mathcal{F}_N, |F \cap P| \leq 1\} \\ & \cup \{F \cup X : F \in \mathcal{F}_N, F \cap P = \{f\}\} \\ & \cup \{F \cup \{x_L\} : F \in \mathcal{F}_N, F \cap P = L \in \mathcal{L}\} \\ & \cup \{F : F \in \mathcal{F}_N, r_M(F \cap P) = 2, F \cap P \notin \mathcal{L}\} \\ & \cup \{F \cup X : F \in \mathcal{F}_N, P \subseteq F\}. \end{aligned}$$

Since a matroid is determined by its collection of cyclic flats, the matroid $\tilde{M}(\overline{A})$ is therefore determined, for a given n and q , by the naming of elements in X and the choice of N, P and f . There is only one choice for N up to isomorphism, and the lemma now follows from the fact that the $\text{Aut}(\text{PG}(n-1, q))$ acts transitively on pairs (P, f) , where P is a plane containing f . \square

We write $\overline{\text{PG}}(n-1, q)$ for any matroid isomorphic to $M(\overline{A})$. Note that $M = \overline{\text{PG}}(n-1, q)$ arises from $N = \text{PG}(n-1, q)$ by adding q^2 new points on a line, spanned by a plane P of M and spanning a single point of P . The following is immediate from the definition and the previous lemma.

Lemma 4.2. *The matroid $\overline{\text{PG}}(n-1, q)$ is $\text{GF}(q)$ -regular.*

We now turn to our second class, which is simpler to analyse. Let \mathbb{F} be a field with a $\text{GF}(q)$ -subfield and let $n \geq 2$. Let $B \in \mathcal{PG}(n, q)$ and $N = \tilde{M}(B)$. Let L_0 be a line of N and $v \in \text{col}_{\mathbb{F}}(B[L_0])$ be a nonzero vector, not parallel to any column of $B[L_0]$. Let $e \notin E(N)$ and $B^+ \in \mathbb{F}^{[n+1] \times (E(N) \cup \{e\})}$ be such that $B^+[E(N)] = B$ and $B^+[e] = v$.

By modularity of N , the matroid $\tilde{M}(B^+)$ is isomorphic to the principal extension of L_0 in N by the element e , and is therefore determined up to isomorphism by n and q (due to transitivity of $\text{Aut}(\text{PG}(n, q))$ on its set of lines). We write $\widehat{\text{PG}}(n-1, q)$ for any matroid isomorphic to the rank- n matroid $\text{si}(\tilde{M}(B^+)/e)$. The following is clear by construction:

Lemma 4.3. *The matroid $\widehat{\text{PG}}(n-1, q)$ is $\text{GF}(q)$ -regular.*

While we have specified these matroids abstractly to emphasise their GF(q)-regularity and the fact that they are well-defined, we will only be interested in their GF(q^2)-representations. We first consider $\overline{\text{PG}}(n-1, q)$. The line X we add is a U_{2, q^2+1} -restriction spanned by an element f of N , together with an element x_{L_0} that is spanned by L_0 but not contained in L_0 . Since there are at most $q^2 + 1$ points on every line in $\text{PG}(n-1, q^2)$, there is only one way to add the points in X given a choice of f and x_{L_0} . By choosing a basis for GF(q^2) n in which L_0 and f correspond to the first three standard basis vectors, we see that $\overline{\text{PG}}(n-1, q)$ has the following as a representation:

$$\overline{A}(n-1, q) = \begin{pmatrix} & x_{L_0} & X - \{x_{L_0}\} & E(N) \\ 1 & \alpha & & \\ \omega & \omega\alpha & & \\ 0 & 1 & A & \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix},$$

where α ranges over GF(q^2) $\setminus \{0\}$, and $A \in \mathcal{PG}(n-1, q)$ is such that A_f is the third standard basis vector.

Now we consider $\widehat{\text{PG}}(n-1, q)$. Let $B \in \mathcal{PG}(n, q)$ be a matrix containing among its columns the standard basis vectors $b_1, \dots, b_{n+1} \in \text{GF}(q)^{n+1}$. If we choose L_0 to be the line spanned by b_1 and b_2 and v to be the vector $b_1 - \omega b_2$, the matroid $\widehat{\text{PG}}(n-1, q)$, obtained by appending v to B and contracting the corresponding element, has the following representation:

$$\widehat{A}(n-1, q) = \begin{pmatrix} (0 + 0\omega)\mathbf{j} & (1 + 0\omega)\mathbf{j} & \dots & (s + t\omega)\mathbf{j} & \dots \\ A & A & \dots & A & \dots \end{pmatrix},$$

where $A \in \mathcal{PG}(n-2, q)$, $\mathbf{j} = (1, \dots, 1)$ denotes the all-ones vector with $\frac{q^{n-1}-1}{q-1}$ entries, and s and t range over GF(q). Note that every vector in GF(q^2) n with all but the first entry in GF(q) is parallel to a column of $\widehat{A}(n-1, q)$.

We have defined $\widehat{\text{PG}}(n-1, q)$ and $\overline{\text{PG}}(n-1, q)$ abstractly, not as GF(q^2)-represented matroids. When we refer to the associated GF(q^2)-represented matroids we will write $M(\widehat{A}(n-1, q))$ and $M(\overline{A}(n-1, q))$.

5. NON-EXAMPLES

Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field \mathbb{F}_0 . For a vector $w \in \mathbb{F}^t$, we write $L(w)$ for the subspace $\text{span}_{\mathbb{F}_0}(\{u, v\})$, where u and

v are the unique \mathbb{F}_0 -vectors so that $w = u + \omega v$. Note that $L(w)$ has dimension 2 if and only if w is not parallel to an \mathbb{F}_0 -vector.

We now define an important class of rank-3 represented matroids that will serve as obstructions to $\text{GF}(q^2)$ -regularity. Let $\mathcal{O}(q)$ denote the set of $\text{GF}(q^2)$ -represented matroids M such that $M \approx M(A \mid G_3)$, where the column set X of A has three elements, $G_3 \in \mathcal{PG}(2, q)$, and $A \in \text{GF}(q^2)^{[3] \times X}$ is a rank-3 matrix such that the three subspaces $L(A_x) : x \in X$ each have dimension 2 and together have trivial intersection.

More geometrically, if $M \in \mathcal{O}(q)$ then \tilde{M} is obtained by extending a projective plane R over $\text{GF}(q)$ by a three-element independent set X so that \tilde{M} is $\text{GF}(q^2)$ -representable and there is no point of R common to the three lines of R spanning the three points of X .

Lemma 5.1. *If $M \in \mathcal{O}(q)$, then \tilde{M} is representable over a field \mathbb{F} if and only if \mathbb{F} has $\text{GF}(q^2)$ as a subfield.*

Proof. Let $M \in \mathcal{O}(q)$ and X, A, G_3 be defined as above. Let $X = \{x_1, x_2, x_3\}$ and $R = M \setminus X$, noting that $\tilde{R} \cong \text{PG}(2, q)$. Each pair of subspaces in $\{L(A_x) : x \in X\}$ meet in dimension 1; let e_i be the unique element of $E(R)$ so that $G_3[e_i] \in \cap_{j \in [3] - \{i\}} (L(x_j))$. Moreover by Lemma 3.2 each pair of columns of A spans a nonzero $\text{GF}(q)$ -vector; for each $i \in [3]$ let f_i be the unique element of $E(R)$ so that $G_3[f_i] \in \text{col}(A[X - \{x_i\}])$. Note that \tilde{M} is a simple rank-3 matroid, that $\tilde{R} \cong \text{PG}(2, q)$, and that the subspaces $L(A_x) : x \in X$ correspond to three lines L_1, L_2, L_3 of \tilde{R} so that $x_i \in \text{cl}_{\tilde{M}}(L_i)$ and $L_1 \cap L_2 \cap L_3 = \emptyset$. Further observe that if $i, j \in [3]$ and $i \neq j$, then $f_i \notin L_j$. Since \tilde{M} is $\text{GF}(q^2)$ -representable it is also representable over all fields with a $\text{GF}(q^2)$ -subfield, so it remains to show that \tilde{M} is not representable over any other fields.

Let \mathbb{F} be a field over which \tilde{M} is representable and assume for a contradiction that \mathbb{F} does not have a $\text{GF}(q^2)$ -subfield. Since \tilde{R} is a minor of \tilde{M} it follows that \mathbb{F} has $\text{GF}(q)$ as a subfield. Let $P \in \mathbb{F}^{[3] \times E(M)}$ be a \mathbb{F} -representation of \tilde{M} ; by Lemma 3.1 we may assume that $P[E(R)]$ is a $\text{GF}(q)$ -matrix and by applying further $\text{GF}(q)$ -row operations and $\text{GF}(q^2)$ -column scalings we may assume (using the fact that $f_i \notin L_j$ for $i \neq j$) that P has the form

$$P = \begin{pmatrix} e_1 & e_2 & e_3 & x_1 & x_2 & x_3 & f_1 & f_2 & f_3 \\ 1 & 0 & 0 & 0 & \alpha_2 & \alpha_3 & s_1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & s_2 & s_4 & \dots \\ 0 & 0 & 1 & \alpha_1 & 1 & 0 & 1 & s_3 & s_5 & \end{pmatrix},$$

where $\alpha_i \in \mathbb{F} - \text{GF}(q)$ for each $i \in [3]$, $s_1 \in \{0, 1\}$ and $s_j \in \text{GF}(q)$ for each $j \in [5]$. Since $r_{\tilde{M}}(x_2, x_3, f_1) = 2$, we have $\alpha_2 + \alpha_3 = s_1$. The lines $\text{cl}_{\tilde{M}}(\{f_2, x_3\})$ and $\text{cl}_{\tilde{M}}(\{x_3, f_2\})$ both intersect L_2 at x_1 , so the vectors $(0, 1 - \alpha_3 s_2, -\alpha_3 s_3)$ and $(0, -\alpha_2 s_4, 1 - \alpha_2 s_5)$ are both parallel to $(0, 1, \alpha_1)$ and thus $\alpha_3 s_3 \alpha_2 s_4 = (1 - \alpha_3 s_2)(1 - \alpha_2 s_5)$. Using $\alpha_3 = s_1 - \alpha_2$, we see that α_2 is a zero of the function

$$p(z) = s_3 s_4 z(s_1 - z) - (1 - s_2(s_1 - z))(1 - s_5 z).$$

Now $p(z)$ is a polynomial in z with coefficients in $\text{GF}(q)$ and degree at most 2. However, $\alpha_2 \notin \text{GF}(q)$ and, since \mathbb{F} has no $\text{GF}(q^2)$ -subfield, α_2 is not a zero of an irreducible quadratic over $\text{GF}(q)$. Therefore $p(z)$ is identically zero. We have $0 = p(0) = 1 - s_1 s_2$, so $s_1 s_2 = 1$; since $s_1 \in \{0, 1\}$ this gives $s_1 = s_2 = 1$. Similarly we have $0 = p(s_1) = s_5 - 1$, so $s_5 = 1$. Therefore $p(z) = z(1 - z)(s_3 s_4 - 1)$, so $s_3 s_4 = 1$. Let $s_3 = t$ and $s_4 = t^{-1}$. Since $r_{\tilde{M}}(\{x_1, x_3, f_2\}) = r_{\tilde{M}}(\{x_1, x_2, f_3\}) = 2$, we have $\alpha_1 = t(1 + \alpha_2^{-1})$ and $\alpha_1 + (1 - \alpha_2)^{-1} = t$. A computation gives $\alpha_2 = (1 + t)^{-1}$, contradicting $\alpha_2 \notin \text{GF}(q)$. \square

We now precisely determine the matrices A which, when appended to a matrix in $\mathcal{PG}(t - 1, q)$, yield a matroid with no $\mathcal{O}(q)$ -minor; these matrices are all essentially restrictions of $\widehat{A}(t - 1, q)$ and $\overline{A}(t - 1, q)$. We also give an alternative characterisation of these matrices in terms of the subspaces $L(x)$ defined as above. This is equivalent to a treatment of the special case of our main theorem where M has a spanning projective geometry restriction.

Lemma 5.2. *Let q be a prime power, $t \geq 3$ be an integer and $G_t \in \mathcal{PG}(t - 1, q)$. If $A \in \text{GF}(q^2)^{[t] \times Y}$ and $M = M(A \mid G_t)$ then the following are equivalent:*

- (1) M has a minor in $\mathcal{O}(q)$;
- (2) $\text{si}(M)$ is not projectively equivalent to a restriction of either $M(\widehat{A}(t - 1, q))$ or $M(\overline{A}(t - 1, q))$;
- (3) there exists a set $Z \subseteq Y$, independent in M , such that $|Z| \in \{2, 3\}$ and the subspaces $L(A_z) : z \in Z$ each have dimension 2 and have trivial intersection.

Moreover, if $t \geq 5$ and (3) is satisfied by a set Z of size 2, then the matroid $M/Z \setminus (Y - Z)$ also has a minor in $\mathcal{O}(q)$.

We call a matrix A satisfying the conditions in this lemma q -bad and if (3) holds with $|Z| = 2$ we call A *strongly q -bad*. Note that property (3), and therefore (strong) q -badness, is invariant under $\text{GF}(q)$ -row equivalence.

Proof of Lemma 5.2: Let b_1, \dots, b_t be the standard basis vectors of $\text{GF}(q)^t$. We showed in Lemmas 4.2 and 4.3 that $\widehat{\text{PG}}(n-1, q)$ and $\overline{\text{PG}}(n-1, q)$ are $\text{GF}(q)$ -regular and in Lemma 5.1 that the matroids in $\mathcal{O}(q)$ are not, so (1) implies (2).

Suppose that (2) holds. Note that (3) and its negation are invariant under $\text{GF}(q)$ -row-equivalence. Let $Y' = \{y \in Y, \dim(L(A_y)) = 2\}$ and $\mathcal{L} = \{L(A_y) : y \in Y'\}$, noting that every $y \in Y - Y'$ is a loop or is parallel to some column of G_t , so $\text{si}(M \setminus (Y - Y')) \cong \text{si}(M)$. If there exist $z_1, z_2 \in Y'$ such that $L(A_{z_1})$ and $L(A_{z_2})$ are skew then $Z = \{z_1, z_2\}$ satisfies (3), so we may assume that Y' contains no such pair.

If all subspaces in \mathcal{L} have a dimension-1 subspace in common, then, by applying $\text{GF}(q)$ -row-operations, we may assume that this subspace is $\text{span}_{\text{GF}(q)}(b_1)$. This gives a matrix representation of $\text{si}(M)$ that is, up to column scaling, a submatrix of $\widehat{A}(t-1, q)$, contradicting (2). We may therefore assume that $\bigcap \mathcal{L}$ is trivial.

Therefore no pair of subspaces in \mathcal{L} are orthogonal but there is no dimension-1 subspace common to all subspaces in \mathcal{L} . It follows routinely that there is some dimension-3 subspace P of $\text{GF}(q)^t$ containing all subspaces in \mathcal{L} , so $r_M(Y') \leq 3$.

If $r_M(Y') \leq 2$ then there is a dimension-2 subspace L_0 of $\text{span}_{\text{GF}(q^2)}(P)$ containing $A[Y']$. By Lemma 3.2, L_0 contains a nonzero $\text{GF}(q)$ -vector v . Let $\{v, w\}$ be a basis for L_0 . After $\text{GF}(q)$ -row-operations we may assume that $\{b_1, b_2, b_3\}$ is a basis for P , that $v = b_3$, and that $w \in \text{cl}_{\text{GF}(q^2)}(\{b_1, b_2\}) - \text{cl}_{\text{GF}(q^2)}(b_2)$. Moreover, after row-scalings over $\text{GF}(q^2)$ we may assume that either $w = b_1$ or $w = b_1 + \omega b_2$. Since $r_M(Y') = 2$ it follows that $\text{si}(M)$ is projectively equivalent to a restriction of $\widehat{A}(t-1, q)$ or $\overline{A}(t-1, q)$, contradicting (2).

If $r_M(Y') = 3$ then let $Z = \{z_1, z_2, z_3\}$ be a basis for Y' . Let $L_i = L(A_{z_i})$ for each $i \in \{1, 2, 3\}$. Since $r_M(Z) = 3$, the lines L_1, L_2, L_3 are not all equal, so we may assume that $L_1 \notin \{L_2, L_3\}$. If L_1, L_2, L_3 have no dimension-1 subspace in common then (3) holds, so we may assume that $L_1 \cap L_2 \cap L_3$ has dimension 1. Moreover we know that there is some other subspace $L_4 = L(A_{z_4}) \in \mathcal{L}$ not containing $L_1 \cap L_2 \cap L_3$, as $\bigcap \mathcal{L}$ is trivial. Now $L_1 \cap L_2 \cap L_4$ and $L_1 \cap L_3 \cap L_4$ are both trivial, and either $\{z_1, z_2, z_4\}$ or $\{z_1, z_3, z_4\}$ has rank 3 in M . Therefore (3) holds.

Finally, suppose that (3) holds. If $|Z| = 2$ then let $Z = \{z_1, z_2\}$. By applying $\text{GF}(q)$ -row-operations if necessary we may assume that $L(z_1) = \text{span}_{\text{GF}(q)}(\{b_1, b_2\})$ and $L(z_2) = \text{span}_{\text{GF}(q)}(\{b_3, b_4\})$. Let X be the set of columns of G_t contained in $\text{span}_{\text{GF}(q)}(L(z_1) \cup L(z_2))$ and

$N = M|(X \cup \{z_1, z_2\})$. We have

$$N \approx M \begin{pmatrix} & z_1 & z_2 & X \\ 1 & 0 & & \\ \alpha_1 & 0 & \dots & \\ 0 & 1 & \dots & \\ 0 & \alpha_2 & & \end{pmatrix},$$

for some $\alpha_1, \alpha_2 \in \text{GF}(q^2) - \text{GF}(q)$, where the matrix contains exactly one column from each parallel class in $\text{GF}(q)^4$. Therefore, N/z_1 is represented by a matrix having a submatrix containing as columns at least one nonzero vector from each parallel class of $\text{GF}(q)^3$, as well as columns parallel to $(0, 1, \alpha_2)^T$, $(-\alpha_1, 1, 0)^T$ and $(-\alpha_1, 0, 1)^T$. Restricting N/z_1 to this submatrix yields a matroid in $\mathcal{O}(q)$. Moreover, if $t \geq 5$ then let X' be the set of columns of t contained in $\text{span}_{\text{GF}(q)}(L(z_1) \cup L(z_2) \cup \{t_5\})$ and let $N' = M|(X' \cup \{z_1, z_2\})$. It is easy to see by a similar argument to the above that $N'/\{z_1, z_2\}$, which is a restriction of $M/Z \setminus (Y - Z)$, has a spanning restriction in $\mathcal{O}(q)$.

If (3) holds for some Z of size 3 but for no 2-element subset of Z , then Z contains three dimension-2 subspaces, all contained in a common dimension-3 subspace, with trivial intersection. This dimension-3 subspace corresponds to a plane P of the spanning $\text{PG}(t-1, q)$ -restriction of M , and clearly $M|(P \cup Z) \in \mathcal{O}(q)$. \square

6. TANGLES

Our tool for constructing minors in $\mathcal{O}(q)$ given a projective geometry minor (rather than a spanning restriction as in Lemma 5.2) is the *tangle*. Tangles were introduced for graphs, and implicitly for matroids, by Robertson and Seymour [10] and were later extended explicitly to matroids [1,3]. The techniques in this section and the next follow [5].

Let M be a matroid and let $\theta \in \mathbb{Z}^+$. A set $X \subseteq E(M)$ is *k -separating in M* if $\lambda_M(X) < k$. A collection \mathcal{T} of subsets of $E(M)$ is a *tangle of order θ* if

- (1) Every set in \mathcal{T} is $(\theta - 1)$ -separating in M and, for each $(\theta - 1)$ -separating set $X \subseteq E(M)$, either $X \in \mathcal{T}$ or $E(M) - X \in \mathcal{T}$;
- (2) if $A, B, C \in \mathcal{T}$ then $A \cup B \cup C \neq E(M)$; and
- (3) $E(M) - \{e\} \notin \mathcal{T}$ for each $e \in E(M)$.

We refer to the sets in \mathcal{T} as *\mathcal{T} -small*. Given a tangle of order θ on a matroid M and a set $X \subseteq E(M)$, we set $\kappa_{\mathcal{T}}(X) = \theta - 1$ if X is contained in no \mathcal{T} -small set, and $\kappa_{\mathcal{T}}(X) = \min\{\lambda_M(Z) : X \subseteq Z \in \mathcal{T}\}$ otherwise. The proof of our first lemma appears in [3]:

Lemma 6.1. *If \mathcal{T} is a tangle of order θ on a matroid M , then $\kappa_{\mathcal{T}}$ is the rank function of a rank- $(\theta - 1)$ matroid on $E(M)$.*

This matroid, which we denote $M(\mathcal{T})$, is the *tangle matroid*. The next lemma is easily proved:

Lemma 6.2. *If N is a minor of a matroid M and \mathcal{T}_N is a tangle of order θ on N , then $\{X \subseteq E(M) : \lambda_M(X) < \theta - 1, X \cap E(N) \in \mathcal{T}_N\}$ is a tangle of order θ on M .*

This tangle is the tangle on M induced by \mathcal{T}_N .

If M is a matroid and k is an integer, then we write $\mathcal{T}_k(M)$ for the collection of $(k - 1)$ -separating sets of M that are neither spanning nor cospanning. For example, if $M \cong \text{PG}(n - 1, q)$ and $n \geq k$, then $\mathcal{T}_k(M)$ is simply the collection of subsets of $E(M)$ of rank at most $k - 2$. Since $3 \frac{q^{n-2}-1}{q-1} < \frac{q^n-1}{q-1}$, no three such subsets have union $E(M)$, and we easily have the following:

Lemma 6.3. *If q is a prime power, $n \in \mathbb{Z}^+$, and $M \cong \text{PG}(n - 1, q)$, then $\mathcal{T}_n(M)$ is a tangle of order n in M .*

If M is a matroid with a $\text{PG}(n - 1, q)$ -minor N , then we write $\mathcal{T}_n(M, N)$ for the tangle of order n in M induced by $\mathcal{T}_n(N)$.

The next result is a slight variation of a lemma from [5].

Lemma 6.4. *Let $k \in \mathbb{Z}^+$, let M be a matroid and let N be a minor of M such that $\mathcal{T}_k(N)$ is a tangle. If $X \subseteq E(M)$ is contained in a $\mathcal{T}_k(M, N)$ -small set, then there is a minor M' of M such that $M'|X = M|X$, M' has N as a minor, and X is contained in a $\mathcal{T}_k(M', N)$ -small set X' such that $E(M') = E(N) \cup X'$ and $\lambda_{M'}(X') = \kappa_{\mathcal{T}_k(M', N)}(X) = \kappa_{\mathcal{T}_k(M, N)}(X)$.*

Proof. Let $b = r_{\mathcal{T}_k(M, N)}(X)$ and let M' be a minimal minor of M such that N is a minor of M , $M|X = M'|X$ and $r_{\mathcal{T}_k(M', N)}(X) = b$. Let $\mathcal{T} = \mathcal{T}_k(M', N)$ and $X' = \text{cl}_{M(\mathcal{T})}(X)$. It remains to show that $E(M') = X' \cup E(N)$. If not, there is some $e \in E(M') - X' \cup E(N)$. Since $\text{cl}_{M'}(X) \subseteq X'$, we know that $M|X$ is a restriction of both M/e and $M \setminus e$. If N is a minor of M/e , and so by choice of M we have $r_{\mathcal{T}_k(M/e, N)}(X) \leq b - 1$. Therefore there is some set $Z \in \mathcal{T}_k(M/e, N)$ such that $\lambda_{M'/e}(Z) \leq b - 1$ and $X \subseteq Z$. Therefore $Z \cup \{e\} \in \mathcal{T}$ and $\lambda_{M'}(Z \cup \{e\}) \leq b$ so $r_{\mathcal{T}}(X \cup \{e\}) = r_{\mathcal{T}}(X)$ and $e \in \text{cl}_{\mathcal{T}}(X)$, a contradiction. The case where N is a minor of $M \setminus e$ is similar. \square

7. USING A TANGLE

Our first lemma allows us to find an affine geometry restriction in a dense $\text{GF}(q)$ -representable matroid M after contracting a subset of

an arbitrary set of bounded size. A stronger qualitative version of this lemma (in which such a restriction is found in M itself) follows from the density Hales-Jewett theorem [2], but the proof of this result is much easier and we obtain a constructive bound.

Lemma 7.1. *Let $\alpha \in \mathbb{R}^+$, q be a prime power, and $n, h, k \in \mathbb{Z}^+$ satisfy $n \geq (2 + k)h + \log_q(2/\alpha)$ and $k \geq 2q^h(1/\alpha - 1)$. If M is a rank- r GF(q)-representable matroid with $r \geq n$ and $\epsilon(M) \geq \alpha |\text{PG}(r - 1, q)|$ then for each rank- hk independent set C in M , there exists $C' \subseteq C$ such that M/C' has an AG(h, q)-restriction.*

Proof. Let (C_1, C_2, \dots, C_k) be a partition of C into sets of size h , and for each $i \in \{0, \dots, k\}$ let $M_i = M/(C_1 \cup \dots \cup C_i)$ and $\delta_i = \epsilon(M_i)/|\text{PG}(r(M_i) - 1, q)|$, noting that $\delta_0 \geq \alpha$ and $\delta_i \leq 1$ for each i . Let $x = \frac{1}{2}q^{-h}$ and let j be maximal such that $j \leq k$ and $\delta_j \geq \alpha(1+x)^j$. If $j = k$ then we have $\delta_k \geq \alpha(1+x)^k > \alpha(1+kx) \geq 1$, a contradiction. Therefore $j < k$, and we have $\delta_j \geq \alpha(1+x)^j$ and $\delta_{j+1} < \alpha(1+x)^{j+1}$.

Let $F = \text{cl}_{M_j}(C_{j+1})$ and \mathcal{F} be the collection of rank- $(h+1)$ flats of M_j containing F ; we have $\epsilon(M_{j+1}) = |\mathcal{F}|$ and $\epsilon(M_j) = \epsilon(M_j|F) + \sum_{H \in \mathcal{F}} (\epsilon(M_j|H) - \epsilon(M_j|F))$. We may assume that $M_j|H \not\cong \text{AG}(h, q)$ for each $H \in \mathcal{F}$, and therefore that $\epsilon(M_j|H) - \epsilon(M_j|F) < q^h$ for each $H \in \mathcal{F}$. Let $r = r(M_j) = n - hk$. Now

$$\begin{aligned} \alpha(1+x)^j \frac{q^r - 1}{q - 1} &\leq \epsilon(M_j) \\ &= \epsilon(M_j|F) + \sum_{H \in \mathcal{F}} (\epsilon(M_j|H) - \epsilon(M_j|F)) \\ &\leq \frac{q^h - 1}{q - 1} + (q^h - 1)\epsilon(M_{j+1}) \\ &< \frac{q^h - 1}{q - 1} + \alpha(q^h - 1)(1+x)^{j+1} \frac{q^{r-h} - 1}{q - 1}. \end{aligned}$$

Simplifying this inequality gives

$$x(q^r - 1) + \frac{q^h - 1}{(1+x)^j \alpha} > (1+x)(q^h + q^{r-h} - 2),$$

and so, using $x > 0$ and $q^h \geq 2$, we have $xq^r + q^h/\alpha > q^{r-h}$. This implies that $q^r < 2q^{2h}/\alpha$, contradicting $r \geq 2h + \log_q(2/\alpha)$. \square

We now combine the previous lemma and the machinery of tangles to show that, given a small restriction of M with given ‘connectivity’ to a large projective geometry minor of M , we can realise the same connectivity to a projective geometry restriction in a minor of M . The

‘qualitative’ version of this lemma, on whose proof ours is based, will appear in [5].

Lemma 7.2. *Let q be a prime power, let $h, a \in \mathbb{Z}^+$ satisfy $a \leq h$ and let $n = 2h(1 + q^{h+a}) + a + 2$. If M is a matroid with a $\text{PG}(n-1, q)$ -minor N and $X \subseteq E(M)$ is a set such that $r_M(X) \leq a$ and $M \setminus X$ is $\text{GF}(q)$ -representable, then there is a minor M' of M and a $\text{PG}(h-1, q)$ -restriction N' of M' such that $E(M') = E(N') \cup X$, $M'|X = M|X$ and $\lambda_{M'}(X) = \kappa_{\mathcal{T}_k(M, N)}(X)$.*

Proof. Let $k = 2q^{h+a}$ and $\alpha = (q^a + 1)^{-1}$, noting that h, k, n and α satisfy the numerical conditions in Lemma 7.1. Let $b = \kappa_{\mathcal{T}_n(M, N)}(X)$. By Lemma 6.4 there is a minor M_1 of M having N as a minor and a $\mathcal{T}_n(M_1, N)$ -small set X_1 containing X such that $E(M_1) = E(N) \cup X_1$ and $\lambda_{M_1}(X_1) = \kappa_{\mathcal{T}_n(M_1, N)}(X) = b$.

Note for each independent set C of N that $\mathcal{T}_{n-|C|}(N/C)$ is a tangle of order $n - |C|$ on N/C . Let C be a maximal independent set of $N \setminus (X \cap E(N))$ so that

- (1) $|C| \leq hk$,
- (2) $M_1|X = (M_1/C)|X$, and
- (3) $\kappa_{\mathcal{T}_{n-|C'|}(M_1/C', N/C')}(X) = b$ for all $C' \subseteq C$.

Let $M_2 = M_1/C$, $N_2 = N/C$, $\mathcal{T} = \mathcal{T}_{n-|C|}(M_2, N_2)$ and $X' = \text{cl}_{M(\mathcal{T})}(X)$.

7.2.1. $|C| = hk$.

Proof of claim: Suppose that $|C| \leq hk - 1$. Since $\kappa_{\mathcal{T}}(X') = b \leq n - hk < n - |C|$, we have $X' \in \mathcal{T}$, so $E(N_2) - X'$ is spanning in N_2 . Further note that $r_{M_2}(X) = a < n - |C|$; let $e \in E(N_2) - X' - \text{cl}_{M_2}(X)$. By choice of C and e , we may assume that X has rank at most $b - 1$ in $\mathcal{T}_{n-|C' \cup \{e\}|}(M_2/e, N_2/e)$ for some $C' \subseteq C$, so there is some set Z such that $C' \cup \{e\} \subseteq Z$, $\lambda_{M_2/e}(Z) \leq b - 1$ and $Z \cap E(N_2/e)$ is not spanning in N_2/e . Therefore $(Z \cup e) \cap E(N_2)$ is not spanning in N_2 and $\lambda_{M_2}(Z \cup \{e\}) \leq b$. It follows that $e \in \text{cl}_{\mathcal{T}}(X) = X'$, a contradiction. \square

Since $X_1 \cap E(N)$ is not spanning in N and N is round, it follows that $r_N(X_1 \cap E(N)) = \lambda_N(X_1 \cap E(N)) \leq \lambda_{M_1}(X_1) = b$. Therefore $n \leq r(M_1|E(N)) \leq n + b$. Now

$$\begin{aligned} \epsilon(M_1 \setminus X_1) &\geq \frac{q^n - 1}{q - 1} - \frac{q^b - 1}{q - 1} \\ &\geq (q^b + 1)^{-1} \frac{q^{n+b} - 1}{q - 1} \\ &\geq \alpha |\text{PG}(r(M_1|E(N)) - 1, q)|. \end{aligned}$$

The matroid $M_1|E(N)$ is a minor of $M \setminus X$ and is therefore GF(q)-representable. Moreover, C is an hk -element independent subset of $E(N)$, so by Lemma 7.1 there is a set $C' \subseteq C$ such that $(M_1|E(N))/C'$ has an AG(h, q)-restriction $(M_1/C')|A$. Let $\mathcal{T}' = \mathcal{T}_{n-|C'|}(M_1/C', N/C')$. Now N/C' is GF(q)-representable and $\epsilon((N/C')|A) = q^h$, so $r_{(N/C')|A} \geq h+1 > b$. Therefore $\kappa_{\mathcal{T}'}(A) \geq \kappa_{\mathcal{T}_{n-|C'|}(N/C')}(A) \geq b$. It follows that $\kappa_{M_1/C'}(X, A) = b$, as otherwise M_1/C' has a b -separation for which neither side is \mathcal{T}' -small.

By Theorem 2.1, there is a minor M'_1 of M_1/C' with $E(M'_1) = X \cup A$, $M'_1|X = (M_1/C')|X = M|X$, $M'_1|A = (M_1/C')|A \cong \text{AG}(h, q)$ and $\lambda_{M'_1}(X) = b$. Since $r(M'_1|A) = h+1 > b$, there is some $e \in A - \text{cl}_{M'_1}(X)$. Contracting e and simplifying yields the required minor M' . \square

Note in the above lemma that, in the special case where M is round we have $\kappa_{\mathcal{T}_k(M, N)}(X) = r_M(X)$; it follows that N' is spanning in M' .

8. AUGMENTING STRUCTURE

We now consider a matroid M and an element $e \in E(M)$ such that $\text{si}(M/e)$ is a restriction of $\widehat{\text{PG}}(r(M) - 2, q)$ or $\overline{\text{PG}}(r(M) - 2, q)$; we essentially argue that M itself either has one of these two structures, or satisfies some constructive condition certifying otherwise. Unfortunately these hypotheses and outcomes are somewhat opaque in the two lemmas that follow; Theorem 9.1 will unify them.

We consider a slight variation of contraction in this section for ease of notation. If e is a nonloop of a represented matroid M , then we let $M//e$ denote the represented matroid M'/e' , where M' is obtained from M by extending e in parallel by an element e' . Thus, e is a loop of $M//e$, and we have $M/e = (M//e) \setminus e$ and $E(M//e) = E(M)$. Note that if $M//e \approx M(A)$ for some \mathbb{F} -matrix A , then $M \approx M(A')$ for some matrix A' obtained by appending a single row to A .

Lemma 8.1. *Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field \mathbb{F}_0 . Let M be a vertically 5-connected \mathbb{F} -represented rank- r matroid and e be a nonloop of M such that $M//e \approx M \binom{u_0 + \omega v_0}{R}$ for some $u_0, v_0 \in \mathbb{F}_0^{E(M)}$ and $R \in \mathbb{F}_0^{[r-2] \times E(M)}$. Then there are matrices $P, Q \in \mathbb{F}_0^{[2] \times E(M)}$ such that $M \approx M \binom{P + \omega Q}{R}$ and either*

- (1) *there is a partition (I, J) of $E(M)$ such that*

$$\text{rank}(R[I]) + \text{rank}(Q[J]) \leq 1,$$

or

- (2) *the matrix*

$$W^+ = \begin{matrix} & S & X & E(M) \\ \begin{matrix} [2] \\ [2] \\ [r-2] \end{matrix} & \begin{pmatrix} I_2 & 0 & -\omega I_2 & P \\ 0 & I_2 & I_2 & Q \\ 0 & 0 & 0 & R \end{pmatrix} \end{matrix}$$

satisfies $\kappa_{M(W^+)}(S \cup X, K) = 4$ for every set $K \subseteq E(M)$ such that $r_M(K) \geq 4$. (Here $|S| = 4$ and $|X| = 2$.)

Proof. Since $M//e \approx M \binom{u_0 + \omega v_0}{R}$, we have $M \approx M \binom{P_1 + \omega Q_1}{R}$ for some $P_1, Q_1 \in \mathbb{F}_0^{[2] \times E(M)}$. Let W^+ be the matrix in (2) with $P, Q = P_1, Q_1$ and let $M^+ = M(W^+)$. Note that $M \approx M^+/X \setminus S$ and $r(M^+) = r + 2$. If (2) does not hold for P_1, Q_1 , then there are sets $Z, K \subseteq E(M^+)$ such that $r_M(K) \geq 4$, with $S \cup X \subseteq Z \subseteq E(M^+) - K$ and $\lambda_{M^+}(Z) \leq 3$. Let $(I, J) = (E(M) \cap Z, E(M) - Z)$.

Note that $r_{M^+}(Z) \geq r_{M^+}(S) = 4$. We have $\lambda_M(I) \leq \lambda_{M^+}(Z) \leq 3$, so vertical 5-connectivity of M gives $\min(r_M(I), r_M(J)) \leq 3$. But $r_M(J) \geq r_M(K) \geq 4$, so $r_M(I) \leq 3$. This gives $r_{M^+}(Z) \leq 5$ and, by vertical 5-connectivity of M , $r_M(J) = r$.

Note that $0 \leq r_{M^+}(J) - r_M(J) \leq r(M^+) - r(M) = 2$. We have $r = r_M(J) = \text{rank} \left(\binom{P_1 + \omega Q_1}{R} [J] \right)$ and $r_{M^+}(J) = \text{rank}(W^+[J])$. By Lemma 3.4, $\binom{P_1 + \omega Q_1}{R} [J]$ is row-equivalent to a matrix $\binom{P' + \omega Q'}{R[J]}$, where

$$\text{rank}(Q') = \text{rank}(W^+[J]) - \text{rank} \left(\binom{P_1 + \omega Q_1}{R} [J] \right) = r_{M^+}(J) - r.$$

Therefore $\binom{P_1 + \omega Q_1}{R}$ is row-equivalent to a matrix $\binom{P + \omega Q}{R}$ where $Q[J] = Q'$. Now $M = M \binom{P + \omega Q}{R}$ and

$$\begin{aligned} 3 &\geq \lambda_{M^+}(Z) \\ &= r_{M^+}(Z) + r_{M^+}(J) - r(M^+) \\ &= (4 + \text{rank}(R[I])) + (r + \text{rank}(Q')) - (r + 2), \\ &= 2 + \text{rank}(R[I]) + \text{rank}(Q[J]) \end{aligned}$$

so $\text{rank}(R[I]) + \text{rank}(Q[J]) \leq 1$. Therefore (1) holds. \square

Lemma 8.2. *Let $\mathbb{F} = \mathbb{F}_0(\omega)$ be a degree-2 extension field of a field \mathbb{F}_0 . Let M be a rank- r , vertically 9-connected \mathbb{F} -represented matroid and e be a nonloop of M . If there are matrices $P_0, Q_0 \in \mathbb{F}_0^{[2] \times E(M)}$ and $R \in \mathbb{F}_0^{[r-3] \times E(M)}$ and a partition (I_0, J_0) of $E(M)$ such that $M//e \approx M \binom{P_0 + \omega Q_0}{R}$, $r_{M//e}(I_0) \leq 2$, $\text{rank}(R[I_0]) \leq 1$ and $Q_0[J_0] = 0$, then there are matrices $P, Q \in \mathbb{F}_0^{[3] \times E(M)}$ such that $M \approx M \binom{P + \omega Q}{R}$ and either*

- (1) M and e satisfy the hypotheses of Lemma 8.1,
- (2) there is a partition (I, J) of $E(M)$ such that $Q[J] = 0$ and $r_M(I) \leq 4$, or

(3) the matrix

$$W^+ = \begin{matrix} & S & X & E(M) \\ \begin{matrix} [3] \\ [3] \\ [r-2] \end{matrix} & \begin{pmatrix} I_3 & 0 & -\omega I_3 & P \\ 0 & I_3 & I_3 & Q \\ 0 & 0 & 0 & R \end{pmatrix} \end{matrix}$$

satisfies $\kappa_{M(W^+)}(S \cup X, K) \geq 5$ for each set $K \subseteq E(M)$ such that $r_M(K) \geq 5$. (Here $|S| = 6$ and $|X| = 3$.)

Proof. By hypothesis, there are matrices $P_1, Q_1 \in \mathbb{F}_0^{[3] \times E(M)}$ such that $M \approx M \begin{pmatrix} P_1 + \omega Q_1 \\ R \end{pmatrix}$, where $P_1 = \begin{pmatrix} u \\ P_0 \end{pmatrix}$ and $Q_1 = \begin{pmatrix} v \\ Q_0 \end{pmatrix}$ for some vectors $u, v \in \mathbb{F}_0^{E(M)}$. Let W^+ be the matrix in (3) with $P, Q = P_1, Q_1$ and let $M^+ = M(W^+)$. As before, we have $M \approx M^+ / X \setminus S$, $r(M^+) = r + 3$ and we may assume that there are sets $Z, K \subseteq E(M^+)$ with $r_M(K) \geq 5$ such that $S \cup X \subseteq Z \subseteq E(M) - K$ and $\lambda_{M^+}(Z) \leq 4$.

Now $\lambda_M(E(M) \cap Z) \leq \lambda_{M^+}(Z) \leq 4$, so vertical 6-connectivity of M gives $\min(r_M(E(M) \cap Z), r(M \setminus Z)) \leq 4$, but $r(M \setminus Z) \geq r_M(K) \geq 5$, so $r_M(E(M) \cap Z) \leq 4$ and thus $r_{M^+}(Z) \leq 7$ and $r_{M^+}(Z) \in \{6, 7\}$. Let $F = \text{cl}_{M^+}(Z)$, let $(I_1, J_1) = (E(M) \cap F, E(M) - F)$ and let $(I, J) = (I_0 \cup I_1, J_0 \cap J_1)$.

We have $r_M(I) \leq (r_{M//e}(I_0) + 1) + r_M(I_1) \leq 3 + 4 = 7$, so by vertical 9-connectivity of M we get $r_M(J) = r$. Therefore $r_{M^+}(J) \geq r$. Moreover $r_{M^+}(J_1) = r(M^+) + \lambda_{M^+}(J_1) - r_{M^+}(F) \leq (r + 3) + 4 - r_{M^+}(F) = r + 7 - r_{M^+}(Z)$, so $r_{M^+}(J_1) \in \{r, r + 1\}$. We consider the two cases separately.

If $r_{M^+}(J_1) = r$ then $r_{M^+}(J) = r$ and $W^+[J]$ is a rank- r matrix with $(r + 3)$ rows, so by Lemma 3.4, $\begin{pmatrix} P_1 + \omega Q_1 \\ R \end{pmatrix}[J]$ is row-equivalent to a matrix $\begin{pmatrix} P' \\ R[J] \end{pmatrix}$ where $P' \in \mathbb{F}_0^{[3] \times J}$. Therefore $\begin{pmatrix} P_1 + \omega Q_1 \\ R \end{pmatrix}$ is row-equivalent to a matrix $\begin{pmatrix} P + \omega Q \\ R \end{pmatrix}$ where $Q[J] = 0$. Now $M \approx M \begin{pmatrix} P + \omega Q \\ R \end{pmatrix}$ and $r_M(I) \leq r_{M^+}(Z) - 3 \leq 4$, so (2) holds.

If $r_{M^+}(J_1) = r + 1$ then $r_{M^+}(F) = 6 = r_{M^+}(S)$ so $F = \text{cl}_{M^+}(S)$. It follows that $R[I_1] = 0$. Also, $W^+[J_1]$ is a rank- $(r + 1)$ matrix with $r + 3$ rows, so by Lemma 3.4 the matrix $\begin{pmatrix} P_1 + \omega Q_1 \\ R \end{pmatrix}[J_1]$ is row-equivalent to a matrix $\begin{pmatrix} P' + \omega Q' \\ R[J_1] \end{pmatrix}$ where $P', Q' \in \mathbb{F}_0^{[3] \times J_1}$ and $Q'[J_1]$ has two zero rows. Therefore $\begin{pmatrix} P_1 + \omega Q_1 \\ R \end{pmatrix}$ is row-equivalent to a matrix $\begin{pmatrix} P + \omega Q \\ R \end{pmatrix}$ where $P, Q \in \mathbb{F}_0^{[3] \times E(M)}$ and $Q[J_1] = Q'$. Since $R[e] = 0$, it follows that $M//e \approx M \begin{pmatrix} P'_0 + \omega Q'_0 \\ R \end{pmatrix}$ for some matrices $P', Q' \in \mathbb{F}_0^{[2] \times E(M)}$ with $\text{rank}(Q'_0[J_1]) \leq \text{rank}(Q') \leq 1$. We may assume (by applying \mathbb{F}_0 -row operations to $P'_0 + \omega Q'_0$ if necessary) that the second row of $Q'_0[J_1]$ is zero. Now $R[I_1] = 0$, so we can scale each column of $\begin{pmatrix} P'_0 + \omega Q'_0 \\ R \end{pmatrix}[I_1]$ to have its

second entry in \mathbb{F}_0 . This yields an matrix $\begin{pmatrix} u_0 + \omega v_0 \\ R' \end{pmatrix}$ where u_0, v_0 are \mathbb{F}_0 -vectors, R' is an \mathbb{F}_0 -matrix, and $M//e \approx M\begin{pmatrix} u_0 + \omega v_0 \\ R' \end{pmatrix}$, so (1) holds. \square

9. THE MAIN THEOREM

By Lemma 5.1, the abstract matroids corresponding to the represented matroids in $\mathcal{O}(q)$ are not $\text{GF}(q)$ -regular. By Lemmas 4.2 and 4.3, restrictions of $\overline{\text{PG}}(r-1, q)$ and $\widehat{\text{PG}}(r-1, q)$ are $\text{GF}(q)$ -regular. The following result, which applies to arbitrary $\text{GF}(q^2)$ -represented matroids, thus has Theorem 1.1 as a corollary.

Theorem 9.1. *Let q be a prime power. If M is a round rank- r $\text{GF}(q^2)$ -represented matroid with a $\text{PG}(12q^{12} + 19, q)$ -minor and no minor in $\mathcal{O}(q)$, then $\text{si}(M)$ is projectively equivalent to a restriction of either $M(\widehat{A}(r-1, q))$ or $M(\overline{A}(r-1, q))$.*

Proof. Let $n = 12q^{12} + 20$ and N be a $\text{PG}(n-1, q)$ -minor of M . Let $\mathcal{T} = \mathcal{T}_n(M, N)$.

If N is spanning in M then, by Lemma 3.1, we have $M \approx M(A \mid G_r)$ for some matrices $G_r \in \mathcal{PG}(r-1, q)$ and A , and the result follows from Lemma 5.2. We may thus assume inductively that there exists $e \in E(M)$ so that N is a minor of M/e and $\text{si}(M/e)$ is a restriction of either $\widehat{\text{PG}}(r-2, q)$ or $\overline{\text{PG}}(r-2, q)$. We consider these cases in two mutually exclusive claims.

9.1.1. *If the matroid $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\widehat{A}(r-2, q))$ then the theorem holds.*

Proof of claim: The matroid M is round (so is vertically 5-connected) and has a $\text{GF}(q^2)$ -representation projectively equivalent to a submatrix of $\widehat{A}(r-2, q)$; it follows that M and e satisfy the hypotheses of Lemma 8.1; Define matrices P, Q, R as in the conclusion of the lemma, so $M \approx M(W)$ where $W = \begin{pmatrix} P + \omega Q \\ R \end{pmatrix}$.

If outcome (1) of Lemma 8.1 holds then there is a partition (I, J) of $E(M)$ so that $\text{rank}(R[I]) + \text{rank}(Q[J]) \leq 1$, so one of these matrices is zero and the other has rank at most 1. If $R[I] = 0$ and $\text{rank}(Q[J]) \leq 1$ then we may perform $\text{GF}(q)$ -row-operations in the first two rows so that only the first row of $Q[J]$ is nonzero and then scale each column in I so that the second entry is in $\{0, 1\}$; since $R[I] = 0$ it follows that $\text{si}(M)$ is projectively equivalent to a restriction of $M(\widehat{A}(r-1, q))$, as required.

If $Q[J] = 0$ and $\text{rank}(R[I]) \leq 1$, then let $A = W[I]$. Note that $r_M(I) \leq 3$. Since $Q[J] = 0$, if the matroid $\text{si}(M(A \mid G_r))$ is projectively

equivalent to a restriction of $M(\widehat{A}(r-1, q))$ or $M(\overline{A}(r-1, q))$ then so is $\text{si}(M)$. Otherwise, A is q -bad (recall Section 5 for a definition). By roundness of M and Lemma 7.2 applied with $a = h = 3$, there is a rank-3 minor M' of M with a $\text{PG}(2, q)$ -restriction N' so that $E(M') = E(N') \cup I$ and $M'|I = M|I$. However M' is obtained from M by contracting and deleting only columns in $W[J]$, so if $G_3 \in \mathcal{PG}(2, q)$ then $M' \approx M(A' | G_3)$ for some matrix A' that is $\text{GF}(q)$ -row-equivalent to A ; the matrix A' is also q -bad, so by Lemma 5.2, the matroid M' has a minor in $\mathcal{O}(q)$.

If outcome (2) of the lemma holds then let W^+ be the given matrix and $M^+ = M(W^+)$, noting that $M \approx M^+/X \setminus S$ and that $W^+[S \cup X]$ is strongly q -bad (with $Z = X$). Let $\mathcal{T}^+ = \mathcal{T}_n(M^+, N)$. Since $\kappa_{M^+}(S \cup X, K) \geq 4$ for each basis or cobasis K of N , it follows that $\kappa_{\mathcal{T}^+}(S \cup X) = 4$ and so, by Lemma 7.2 applied with $a = 4$ and $h = 5$, M^+ has a minor M' with a $\text{PG}(4, q)$ -restriction N' so that $E(M') = E(N') \cup (S \cup X)$ and $M'|(S \cup X) = M|(S \cup X)$. Similarly to the previous case, we have $M' \approx M(B | G_5)$ for some $G_5 \in \mathcal{PG}(4, q)$ and some matrix B that is $\text{GF}(q)$ -row-equivalent to $W^+[S \cup X]$ and hence strongly q -bad. By Lemma 5.2, the matroid $M'/X \setminus S$, which is a minor of M , has a minor in $\mathcal{O}(q)$, again a contradiction. \square

9.1.2. *If the matroid $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\overline{A}(r-2, q))$ but not to a restriction of $M(\widehat{A}(r-2, q))$ then the theorem holds.*

Proof of claim: Since M is vertically 9-connected. Since $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\overline{A}(r-2, q))$, it is easy to see that M and e satisfy the hypotheses of Lemma 8.2. (The required partition (I_0, J_0) is induced by the line L_0 and its complement in the column set of $\overline{A}(r-2, q)$.) If outcome (1) of the lemma holds then $\text{si}(M/e)$ is projectively equivalent to a restriction of $M(\widehat{A}(r-2, q))$, a contradiction. Therefore (2) or (3) holds. Let $M \approx M(W)$ where $W = \binom{P+\omega Q}{R}$ as in the lemma.

Suppose that (2) holds, and let (I, J) be the associated partition of $E(M)$. If $\text{si}(M((W[I] | G_r)))$ is projectively equivalent to a restriction of $M(\widehat{A}(r-1, q))$ or $M(\overline{A}(r-1, q))$ then, as $W[J]$ is a $\text{GF}(q)$ -matrix, so is $\text{si}(M)$. Therefore we may assume that this is not the case, so $W[I]$ is q -bad. By roundness of M we have $\kappa_{\mathcal{T}}(I) = r_M(I) \leq 4$, so Lemma 7.2 with $a = h = 4$ gives a rank-4 minor M' of M with a $\text{PG}(3, q)$ -restriction N' satisfying $E(M') = E(N') \cup I$ and $M'|I = M|I$. Now $E(M) - E(M') \subseteq J$ and so $M' \approx M(B | G_4)$ for some $G_4 \in \mathcal{PG}(3, q)$ and some matrix B that is $\text{GF}(q)$ -row-equivalent to

$W[I]$ and hence q -bad. Lemma 5.2 implies that M' has a minor in $\mathcal{O}(q)$, a contradiction.

Finally, suppose that (3) holds. Let W^+ be the matrix given and let $M = M(W^+)$, noting that $M = M^+ / X \setminus S$. Let $\mathcal{T}^+ = \mathcal{T}_n(M^+, N)$. Since $\kappa_{\mathcal{M}^+}(S \cup X, K) \geq 5$ for each basis or cobasis K of N , we have $\kappa_{\mathcal{T}^+}(S \cup X) \geq 5$. By Lemma 7.2 with $a = h = 6$ there is a minor M' of M^+ and a $\text{PG}(5, q)$ -restriction N' of M' so that $E(M') = E(N') \cup X \cup S$, $M'|_{(X \cup S)} = M|_{(X \cup S)}$ and $\lambda_{M'}(X \cup S) \geq 5$, from which it follows that $6 \leq r(M') \leq 7$.

Since $W^+[E(M)]$ is a $\text{GF}(q)$ -matrix, we have $M' \approx M(B \mid G)$, where B is obtained by appending a row of zeroes above $W^+[S \cup X]$ and G is a $\text{GF}(q)$ -representation of $N' \cong \text{PG}(5, q)$ with 7 rows. (If $r(M') = 6$ then the first row of G is also zero). Let v_0, \dots, v_6 denote the row vectors of G , so $M' / X \setminus S \approx M(W')$, where

$$W' = \begin{pmatrix} v_0 \\ v_1 + \omega v_4 \\ v_2 + \omega v_5 \\ v_3 + \omega v_6 \end{pmatrix}.$$

For each $i \in \{0, \dots, 6\}$ let G^i be the matrix obtained by removing the i th row of G . Since $\tilde{M}(G) \cong \text{PG}(5, q)$, there is some $i \in \{0, \dots, 6\}$ so that $\tilde{M}(G^i) \cong \text{PG}(5, q)$. Furthermore, unless $v_0 = 0$ we may choose i to be nonzero. If $v_0 = 0$ then, since $\tilde{M}(G^0) \cong \text{PG}(5, q)$, every vector in $\text{GF}(q^2)^4$ with first component zero is a $\text{GF}(q)$ -multiple of some column of W' , so $\text{si}(M(W')) \cong \text{PG}(2, q^2)$ and $M' / X \setminus S$ clearly has a restriction in $\mathcal{O}(q)$, a contradiction.

Otherwise, we can choose i nonzero such that $\tilde{M}(G^i) \cong \text{PG}(5, q)$. We will suppose that $i = 6$; the other cases are similar. Since G^6 contains a column from every parallel class in $\text{GF}(q)^5$, there is some $f \in E(N')$ so that $G^6[f]$ has all entries zero except its v_3 -entry which is nonzero. Therefore $W'[f]$ has all entries zero except its last entry which is nonzero. Now consider a representation W'' of $M(W')/f$ given by removing the f -column and last row from W' . Since the matrix with rows v_0, v_1, v_2, v_4, v_5 has a column in every parallel class in $\text{GF}(q)^5$, it follows that W'' contains a column from every parallel class in $\text{GF}(q^2)^3$, and so $\text{si}(M(W'')) \cong \text{PG}(2, q^2)$ and $M(W'')$ has a restriction in $\mathcal{O}(q)$, a contradiction. \square

The result now follows from the two claims. \square

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